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Research Paper

Cumulative prospect theory and mean–variance analysis: a rigorous comparison

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ABSTRACT

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types of derivatives, such as call options, the restriction of asset allocations to the mean–variance efficient frontier produces sizable losses in various respects, including decreases in expected returns and expected utility. We are able to explain these differences by CPT's preference for positive skewness, which is not accounted for by optimizing CPT along the mean–variance efficient frontier.

Keywords: cumulative prospect theory (CPT); mean–variance (MV) analysis; portfolio selection; adaptive simplicial grid refinement; proximity measures for portfolios.

1 INTRODUCTION

Since the publication of the seminal work of Markowitz (1952), the mean–variance (MV) model has been the main paradigm for addressing asset allocation issues in finance, in theory and practice, both as a prescriptive model of how investors should invest and as a model for describing investors' observed behavior. The solutions of this model are well understood, and efficient algorithms for computing these solutions are readily available. However, during the course of the behavioral revolution in economics, Kahneman and Tversky (1979) amassed evidence that the decisions of actual investors depart from the MV model in many ways. Investors consider the deviations of their terminal wealth as gains and losses starting from a reference level, and they react differently to gains than to losses. Further, investors systematically distort probabilities. This finding raises the question of how asset allocations can be determined for prospect theory (PT). The PT utility function is S-shaped and nonsmooth; therefore, the computation of PT-based asset allocations is nontrivial. Moreover, the original concepts of PT violate first-order stochastic dominance. Later, Tversky and Kahneman (1992) resolved this difficulty by developing cumulative prospect theory (CPT).

The main objectives of our paper are as follows. We propose a numerical optimization approach that can be used to solve portfolio selection problems that involve several assets and employ objective functions from CPT. Implementing the suggested algorithm, we conduct a numerical study based on a real-life data set for asset returns and a widely used data set that incorporates the parameters of PT, including probability distortions. Our goal is to numerically assess the robustness of the following theoretical results, which have been obtained by different authors for various assumptions regarding the distribution of the asset returns.

- The asset allocations obtained by solving the portfolio optimization problem with a CPT objective are located along the MV efficient frontier. Thus, maximizing a CPT objective along the efficient MV frontier provides a good approximation of the solution of the portfolio selection problem with a CPT objective (see Levy and Levy 2004; Pirvu and Schulze 2012).

- Investors with a CPT objective function prefer asset allocations with a positive skewness (see Barberis and Huang 2008; Bernard and Ghossoub 2010).

Analytical solutions in PT asset allocations can only be generated for very simple cases due to the nondifferentiability and nonconcavity of the components of PT. Thus, to determine what asset allocations based on PT look like, one must adopt a computational approach. As a way of implementing this approach, Levy and Levy (2004) suggested the simple solution of restricting one's attention to MV-optimal portfolios and then selecting the portfolio with the highest prospect utility. As Levy and Levy (2004) demonstrated, this procedure is completely acceptable for normally distributed returns. In a situation involving these types of returns, nothing is lost by restricting asset allocations to the MV efficient frontier. In a situation involving a risk-free asset and several risky assets, Pirvu and Schulze (2012) generalize the results of Levy and Levy (2004) for the class of elliptically symmetric distributions of risky assets. These papers allow for unlimited short sales and present an analytical solution that is essentially equivalent to maximizing the CPT objective function along the MV frontier. However, many asset allocation problems involve nonelliptically distributed returns, given that even at the level of broad indexes the returns of stocks, bonds and commodities typically have fat tails and are skewed. Moreover, many investors currently include derivatives in their portfolios, which renders the assumption of elliptically distributed returns unreasonable and unrealistic.

Another interesting analytical result was obtained by Barberis and Huang (2008). In an asset-pricing framework, these authors explore the effects of skewness on optimal portfolio choice. They assume a normal distribution of the risky assets and add a small, independent and positively skewed security to these assets. These authors find that, unlike MV investors, CPT-based investors have a definitive preference for positively skewed securities, apparently because of probability weighting. This effect has also been analyzed by Bernard and Ghossoub (2010) for the two-asset case.

Although Levy and Levy (2004) and Pirvu and Schulze (2012) allow for unlimited short sales, in this paper we assume that short sales are forbidden. As our numerical results regarding the case of an empirical discrete distribution based on normally distributed returns demonstrate, this difference in the model formulation does not create the observed differences between CPT and MV. We find that even with short sales forbidden, in the case of an empirical distribution obtained via sampling and subsequent clustering from normally distributed returns the CPT-optimal portfolios are located on or very close to the efficient MV frontier. Moreover, this restriction makes our results more representative of real-life applications, such as assessments of the investment choices of unsophisticated private investors or the structure of pension funds.

We compare the optimal portfolios obtained for CPT optimizations with those obtained by maximizing a CPT objective function along the MV frontier of Markowitz (1952). The computations have been performed using a benchmark data set of monthly returns that contains eight indexes representing different asset classes. This data set, which is available from the Bloomberg database, is typically used to compute strategic asset allocations in various contexts, eg, for pension funds or private investors.

Throughout this paper, we work with the piecewise power value function specified by Tversky and Kahneman (1992). Instead of varying the parameters of this function in an artificial fashion, we take the parameters of the forty-eight subjects obtained via parameter-free measurement by Abdellaoui *et al* (2007); for more information, see Abdellaoui *et al* (2007, Appendix C). This data also allows us to account for probability distortion in conducting our numerical tests.

To compare the optimal portfolios obtained by CPT optimization with the portfolios that result from maximizing the CPT objective function along the MV efficient frontier, we employ three frequently used indexes based on objective function values and differences in the certainty equivalents. In addition, we compute the distances between the optimal CPT portfolios and the MV efficient frontiers.

The computations were performed using an empirical discrete distribution that was computed via k-means clustering from the historical data. In addition, to test the effects of larger deviations from the normal distribution assumption and the skewness-loving investor attitude under CPT, we added a call option to the empirical distribution data. To compare our results with those of the theoretical literature, we also worked with a test data set that was generated from a multivariate normal distribution with the same expected value vector and covariance matrix as the original empirical distribution. An additional empirical discrete distribution created through sampling and subsequent clustering was computed for this purpose.

The rest of the paper is structured as follows. In Section 2, we formulate the CPT portfolio optimization problem, discuss it from a numerical point of view and present our proposal for an algorithm that can be used to numerically solve this type of problem. Section 3 presents the three proximity indexes used to compare the optimal portfolios obtained via the alternative approaches. Section 4 describes and discusses the data sets utilized in our numerical experiments. In Section 5, we present the numerical results we obtained for the CPT–MV relationship. Section 6 concludes the paper.

2 SOLVING BEHAVIORAL PORTFOLIO CHOICE PROBLEMS NUMERICALLY

2.1 Problem formulations

The basic portfolio optimization model, based on CPT objective function maximization, can be formulated as follows:

$$\left.\begin{aligned} \max_{\lambda} W(\lambda) := V(\xi^{\mathrm{T}}\lambda), \\ \mathbf{1}^{\mathrm{T}}\lambda = 1, \\ \lambda \geqslant 0, \end{aligned}\right\} \tag{2.1}$$

where ξ is the vector of asset returns, λ_i is the weight of the ith asset in the portfolio, $i = 1, \dots, n$, and $\mathbf{1}^{\mathrm{T}} = (1, \dots, 1)$. V is an objective function that corresponds to the CPT. Short sales are excluded in this model due to the nonnegativity constraint on the asset weights. The optimal solution to the above problem is denoted by λ^*.

Throughout this paper, we consider finitely distributed asset returns, which are given by a table of scenarios:

$$\begin{pmatrix} \xi^1, \dots, \xi^S \\ p_1, \dots, p_S \end{pmatrix}; \quad p_s > 0 \text{ for all } s \quad \text{and} \quad \sum_{s=1}^{S} p_s = 1. \tag{2.2}$$

To formulate the objective function of (2.1), we first introduce a value function that plays a role similar to that of a utility function in expected utility theory. In this paper, we will employ the piecewise power value function of Tversky and Kahneman (1992), which can be formulated as follows:

$$v(x) = \begin{cases} (x - \mathrm{RP})^{\alpha^+} & \text{if } x \geqslant \mathrm{RP}, \\ -\beta(\mathrm{RP} - x)^{\alpha^-} & \text{if } x < \mathrm{RP}, \end{cases} \tag{2.3}$$

where RP is the reference point, α^+ and α^- are the risk aversion parameters and β denotes the loss-aversion parameter. The value function (2.3) is consistent with the experiments reported in Tversky and Kahneman (1992) only if the conditions $0 < \alpha^+, \alpha^- \leqslant 1$ and $\beta > 1$ are assumed to hold. Tversky and Kahneman (1992) found that the median parameter values are $\alpha^+ = \alpha^- = 0.88$ and $\beta = 2.25$. According to the value function (2.3), investors evaluate their gains and losses with respect to an RP. For $0 < \alpha^+, \alpha^-$, in the gain domain $\{x \mid x > \mathrm{RP}\}$ investors are risk averse, whereas in the loss domain $\{x \mid x < \mathrm{RP}\}$ risk-seeking behavior prevails, since the function v is strictly concave in the gain domain and strictly convex in the loss domain. The function is steeper in the loss domain than in the gain domain (provided that $\alpha^+ = \alpha^-$ holds) due to the requirement that $\beta > 1$.

The probability distortion function w models observed investor behavior by overweighing small probabilities and underweighing large probabilities. We will use the original probability weighting function of Tversky and Kahneman (1992), which can be formulated as follows:

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}, \tag{2.4}$$

with $0 < \gamma \leqslant 1$. According to the experiments of Tversky and Kahneman (1992), the median value for γ is $\gamma = 0.65$. However, as observed by Rieger and Wang (2006) and Ingersoll (2008), for γ values that are less than 0.278, this function loses the desired strict monotonic increasing property.

In CPT, the probability weighting is applied to the cumulative probability distribution rather than the individual probabilities. Therefore, to formulate the objective function for CPT for a fixed λ, the vector of portfolio returns $((\xi^1)^{\mathrm{T}}\lambda, \dots, (\xi^S)^{\mathrm{T}}\lambda)$ is first sorted in increasing order. Let $((\eta^1)^{\mathrm{T}}\lambda, \dots, (\eta^S)^{\mathrm{T}}\lambda)$ denote the sorted vector, for which

$$(\eta^1)^{\mathrm{T}}\lambda \leqslant \cdots \leqslant (\eta^t)^{\mathrm{T}}\lambda \leqslant \mathrm{RP} \leqslant (\eta^{t+1})^{\mathrm{T}}\lambda \leqslant \cdots \leqslant (\eta^S)^{\mathrm{T}}\lambda \tag{2.5}$$

holds, and let $(\bar{p}_1, \dots, \bar{p}_S)$ be the correspondingly sorted vector of probabilities. Given these specifications, the objective function is computed according to the following equation:

$$W(\lambda) := V(\xi^{\mathrm{T}}\lambda) = \sum_{i=1}^{S} \pi_i v((\eta^i)^{\mathrm{T}}\lambda), \tag{2.6}$$

where the weights π_i are determined using the probability weighting functions w^+ and w^-,

$$\pi_1 = w^-(\bar{p}_1) \quad \text{and} \quad \pi_i = w^-(\bar{p}_1 + \cdots + \bar{p}_i) - w^-(\bar{p}_1 + \cdots + \bar{p}_{i-1}) \tag{2.7}$$

for $2 \leqslant i \leqslant t$, and

$$\pi_S = w^+(\bar{p}_S) \quad \text{and} \quad \pi_j = w^+(\bar{p}_j + \cdots + \bar{p}_S) - w^+(\bar{p}_{j+1} + \cdots + \bar{p}_S) \tag{2.8}$$

for $t < j \leqslant S - 1$; and where the probability weighting functions are defined as follows:

$$w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}} \quad \text{and} \quad w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}, \tag{2.9}$$

with $0 < \delta \leqslant 1$ and $0 < \gamma \leqslant 1$. According to this construction, both tails of the probability distribution are distorted.[1]

[1] For a detailed presentation of the various aspects of PT, see, for example, Hens and Bachmann (2008) or Wakker (2010); for an integrated presentation of MV, the capital asset pricing model (CAPM) and CPT, see Levy (2012).

The maximization of a CPT objective function V along the MV efficient frontier (CPT(MV) approach) can be formulated as determining a solution of the following optimization problem:

$$\left.\begin{aligned} \max_{\lambda} \quad & V(\xi^{\mathrm{T}}\lambda), \\ & \lambda \in \Lambda^*_{\mathrm{MV}}, \end{aligned}\right\} \tag{2.10}$$

with Λ^*_{MV} denoting the set of portfolios along the MV efficient frontier. A basic approach for combining expected utility maximization and MV analysis, both in theory and practice, consists of maximizing expected utility along the MV frontier. In the above formulation, the utility function is replaced by the PT value function and, according to CPT, rank-dependent expected utility is computed.

Let λ^*_{MV} denote an optimal solution to the above problem (2.10). Given that λ^* represents an optimal solution to our PT optimization problem (2.1), the inequality $V(\xi^{\mathrm{T}}\lambda^*_{\mathrm{MV}}) \leqslant V(\xi^{\mathrm{T}}\lambda^*)$ must clearly hold due to the relation

$$\Lambda^*_{\mathrm{MV}} \subset \{\lambda \mid \mathbf{1}^{\mathrm{T}}\lambda = 1,\ \lambda \geqslant 0\}.$$

Levy and Levy (2004) demonstrated that the optimal solutions of the two optimization problems (2.1) and (2.10) coincide under the following conditions: the asset returns are normally distributed and a CPT objective function is maximized.

2.2 Numerical solution approaches

To solve the optimization problem (2.10), corresponding to the maximization of a (C)PT objective function along the MV frontier, the following straightforward numerical method can be utilized: First, generate a sufficiently fine mesh along the MV efficient frontier. Second, compute the CPT objective function value for each of the corresponding portfolios. Finally, select a portfolio with the highest objective value. For details regarding this method, see, for example, Kroll *et al* (1984), Levy and Levy (2004) and De Giorgi and Hens (2009).

In our computational experiments, we proceeded as follows. First, we computed the interval of definition $[\mu_{\min}, \mu_{\max}]$ for the efficient frontier: we computed $\mu_{\min}$ by minimizing the portfolio variance $\lambda^{\mathrm{T}}\Sigma\lambda$, and we computed $\mu_{\max}$ by maximizing the portfolio expected value $r^{\mathrm{T}}\lambda$, both over the unit simplex. Second, we computed an equidistant subdivision $\mu_0 = \mu_{\min} < \mu_1 < \cdots < \mu_N = \mu_{\max}$ of $[\mu_{\min}, \mu_{\max}]$, with $N = 1000$. Subsequently, for each of the subdivision points μ_k we solved the related MV quadratic optimization problem:

$$\min\{\lambda^{\mathrm{T}}\Sigma\lambda \mid r^{\mathrm{T}}\lambda \geqslant \mu_k,\ \mathbf{1}^{\mathrm{T}}\lambda = 1,\ \lambda \geqslant 0\},$$

yielding the frontier portfolios $\lambda^0_{\mathrm{F}}, \lambda^1_{\mathrm{F}}, \dots, \lambda^N_{\mathrm{F}}$. Finally, from among these portfolios, we selected a portfolio with the highest CPT objective value.

TABLE 1 Examples with two assets and three scenarios.

(a) Example 1

Probability	Asset 1	Asset 2
0.5	−0.035	−0.01
0.2	0.03	−0.07
0.3	−0.025	0.01

(b) Example 2

Probability	Asset 1	Asset 2
0.5	0.035	0.005
0.2	−0.03	0.09
0.3	0.025	−0.01

In (a) expected asset returns are negative, whereas in (b) they are positive.

The direct solution of the portfolio selection problem (2.1) turns out to be quite difficult in terms of numerical optimization for the following reasons.

The PT value function v is generally a nonsmooth function that is neither concave nor convex. Even if the value function v is smooth, the CPT objective function V remains nonsmooth because the computation of this function involves the sorting of the portfolio-return realizations, and this order will depend on λ.

To illustrate the sources of the numerical difficulties outlined above, we consider two artificial examples with two assets and three scenarios (see Table 1).

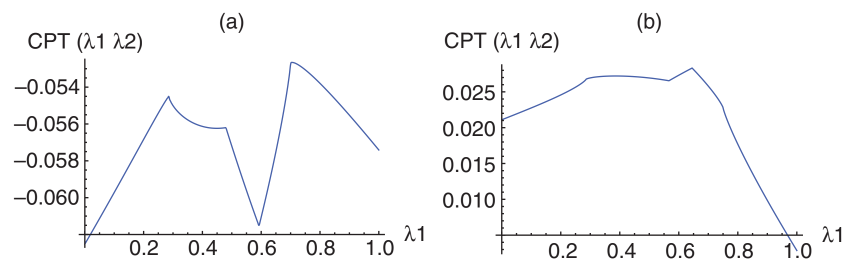
The graphs of the CPT objective function W over the feasible domain are shown in Figure 1, where we display

$$W(\lambda_1, 1-\lambda_1) \quad \text{for } 0 \leqslant \lambda_1 \leqslant 1.$$

In both cases, nonconcavity and nonsmoothness are clearly recognizable.

Observe that the feasible domain of our optimization problem (2.1) is the unit simplex. For economic equilibrium problems involving the unit simplex, a widely used method of computing fixed points is based on simplicial partitions; for a detailed presentation of this type of algorithm, see Dang (1995). We choose to use a method of this type to solve the portfolio optimization problem (2.1). We have chosen the method developed by Kuhn (1968) and Eaves (1971) and have implemented this approach using an adaptive simplicial refinement procedure that is executed within a multistart framework.

FIGURE 1 The graphs of the CPT objective functions corresponding to the examples in Table 1.



PT parameter settings: RP = 0, $\alpha^+ = \alpha^- = 0.88$, $\beta = 2.25$, $\gamma = 0.65$.

Geometrically, the unit simplex

$$\mathcal{U} = \left\{ x \;\middle|\; x = \sum_{k=1}^{n} \mu_k \mathrm{e}^{(k)},\ \sum_{k=1}^{n} \mu_k = 1,\ \mu_k \geqslant 0 \text{ for all } k \right\}$$
$$= \left\{ x \;\middle|\; \sum_{j=1}^{n} x_j = 1,\ x_j \geqslant 0 \text{ for all } j \right\} = \mathbf{conv}\{\mathrm{e}^{(1)}, \dots, \mathrm{e}^{(n)}\}$$

is an $(n-1)$-dimensional simplex in $\mathbb{R}^n$, where the unit vectors $\mathrm{e}^{(1)}, \dots, \mathrm{e}^{(n)}$ are its vertexes and **conv** means building the convex hull of the points. In general, an $(n-1)$-dimensional simplex in $\mathbb{R}^n$ is defined as

$$\mathcal{S} = \left\{ x \;\middle|\; x = \sum_{k=1}^{n} \mu_k v^{(k)},\ \sum_{k=1}^{n} \mu_k = 1,\ \mu_k \geqslant 0 \text{ for all } k \right\} = \mathrm{conv}\{v^{(1)}, \dots, v^{(n)}\},$$

where $v^{(1)}, \dots, v^{(n)}$ are affinely independent vectors. $v^{(1)}, \dots, v^{(n)}$ are called the vertexes of the simplex $\mathcal{S}$, and $\mu_1, \dots, \mu_n$ are the (uniquely determined) barycentric coordinates of $x \in \mathcal{S}$. Note that for elements of the unit simplex barycentric and spatial coordinates coincide.

The main idea behind the simplicial subdivision methods is the following. Simplicial partitions are carried out for the unit simplex $\mathcal{U}$; via barycentric coordinates, these imply simplicial subdivisions for every $(n-1)$-dimensional simplex $\mathcal{S} \subset \mathbb{R}^n$. In our case, the simplices $\mathcal{S}$ will be successively refined subsimplices of the unit simplex $\mathcal{U}$.

A simplex is called regular if all of its edges have the same length. A regular simplicial grid (also called a mesh or a uniform simplicial grid) with grid constant $\mathrm{gc} \in \mathbb{N}$, $\mathrm{gc} > 0$ (also called grid resolution), results from partitioning the unit simplex $\mathcal{U}$ into congruent regular subsimplices, such that the edges of U become partitioned into gc equal pieces. Algebraically, the set of grid points of a regular grid over $\mathcal{U}$ is the following:

$$\mathcal{G}_{\mathrm{gc}} := \left\{ x \;\middle|\; x = \frac{1}{\mathrm{gc}}(k_1, \dots, k_n)^{\mathrm{T}},\ \sum_{i=1}^{n} k_i = \mathrm{gc},\ k_i \in \mathbb{N},\ k_i \geqslant 0 \text{ for all } i \right\}.$$

The regular grid is computed by employing compositions, which are n-tuples $(k_1, \dots, k_n)$ with $\sum_{i=1}^{n} k_i = \mathrm{gc}$, $k_i \in \mathbb{N}$, $k_i \geqslant 0$ for all i. For setting up a list of all compositions, we utilized the recursive algorithm of Kuhn (1968).

The number of grid points in a regular simplicial grid in $\mathbb{R}^n$ with grid constant gc clearly equals the number of compositions, thus yielding the number

$$N_{\mathrm{gc}} := \binom{\mathrm{gc} + n - 1}{\mathrm{gc}}.$$

The crucial ingredient of the simplicial refining procedure is the algorithm for solving the following problem: given a point $p \in \mathcal{U}$, determine a subsimplex corresponding to the current simplicial grid, which contains p. For this task, we have implemented the algorithm developed by Kuhn (1968) and Eaves (1971).

Let gc be the grid constant that we choose as a power of 2. Before starting up the main procedure, the two sets of grid points $\mathcal{G}_{\mathrm{gc}}$ and $\mathcal{G}_{\overline{\mathrm{gc}}}$ of the unit simplex $\mathcal{U}$ are computed, with $\overline{\mathrm{gc}} := \frac{1}{2}\mathrm{gc}$.

Let the starting simplex be $S_0 = \mathrm{conv}\{v^{(1)}, \dots, v^{(n)}\} \subseteq \mathcal{U}$, k the iterations counter and $k_{\max}$ the number of refinement steps. The basic simplicial refining procedure that we applied for solving the optimization problem (2.1) consists of the steps detailed in Box 1.

We have combined the basic algorithm with simulation techniques by embedding the basic algorithm into an outer loop. An outer loop is needed because the objective function to be maximized is nonconcave; the resulting overall algorithm is a multistart random search method (see, for example, Törn and Žilinskas 1989).

Let s be the iterations counter, and $s_{\max}$ be the maximum number of the outer loop steps. The overall algorithm works as detailed in Box 2, with the initial setting $s := 1$ and $W_{\max} = -\infty$. Set $s := 1$ and $W_{\max} := -\infty$.

In our computational experiments, we have chosen $\mathrm{gc} = 8$ for the grid resolution and employed $k_{\max} = 4$ adaptive refinement steps in the basic procedure. This resulted in a terminal mesh size of 0.002, which was sufficiently small in our computational experiments. The outer simulation loop involved $s_{\max} = 20$ starts, with a sample size of $N = 1000$ used to compute the starting point for each of these individual starts of the basic procedure.

BOX 1 Basic simplicial refining procedure.

- *Step 0*: *initialization.* Choose $S_1 := S_0$ as the starting simplex, and set $k := 1$.
- *Step 1*: *choosing the best grid point.* With the elements of $\mathcal{G}_{\text{gc}}$ as barycentric coordinates, compute the grid points of a simplex grid for the current simplex S_k and choose a grid point x^* with the highest objective function value across this grid. If $k = k_{\max}$, then STOP and deliver x^* as the approximate solution of (2.1).
- *Step 2*: *determining a subsimplex of $\mathcal{U}$.* By employing the method of Kuhn (1968) and Eaves (1971), determine the vertexes of a subsimplex of $\mathcal{U}$ corresponding to the coarser grid $\mathcal{G}_{\bar{\text{gc}}}$, which contains x^*.
- *Step 3*: *determining a subsimplex of S_k.* Via barycentric coordinates, compute the vertexes of the corresponding subsimplex of S_k. This subsimplex becomes the current simplex S_{k+1}; set $k := k + 1$ and repeat the procedure by carrying out Step 1.

BOX 2 Outer loop for the refining procedure.

- *Step 1*: *simulation.* Generate N uniformly distributed points in the unit simplex $\mathcal{U}$ and on its boundary using Rubinstein (1982, Algorithm 2), and determine a point $\tilde{\lambda}$ with the highest objective value over this set of points.
- *Step 2*: *determining a subsimplex.* Compute the vertexes $\{v^{(1)}, \dots, v^{(n)}\}$ of a subsimplex of $\mathcal{U}$ with grid resolution $\overline{\text{gc}}$, which contains $\tilde{\lambda}$.
- *Step 3*: *applying the basic subdivision method.* Next start up the basic simplicial subdivision method, with $S_0 = \mathbf{conv}\{v^{(1)}, \dots, v^{(n)}\}$ as the starting simplex. This method delivers the best vector found, x^*.
- *Step 4*: *updating.* If for the objective value in (2.1) $W(x^*) > W_{\max}$ holds, then set $\lambda^* := x^*$ and $W_{\max} := W(x^*)$. If $s = s_{\max}$, then STOP and deliver λ^* as the approximate solution of (2.1); otherwise, repeat the procedure by carrying out Step 1.

As mentioned above, the number of grid points in a uniform simplicial grid over the unit simplex $\mathcal{U} \subset \mathbb{R}^n$, with grid constant gc, is

$$\binom{\text{gc} + n - 1}{\text{gc}},$$

which is a polynomial of n having grade gc $-$ 1. Since the computational time of all other operations in the algorithm (including the algorithm of Kuhn (1968) and Eaves (1971)) also has a polynomial order regarding the number of variables n of the optimization problem (2.1), we conclude that, regarding computational complexity, the computational time of the proposed algorithm has a polynomial order of complexity with respect to n.

In comparison, let us consider a uniform rectangular grid, obtained by subdividing all edges of an n-dimensional hypercube into gc equidistant subintervals. The number of grid points in such a rectangular grid is $(\text{gc}+1)^n$, which is an exponential function of n. From the point of view of computational complexity, this means that the rectangular version of the adaptive grid search method has an exponential order of complexity regarding the dependence of the computational time from the dimension n. Thus, in comparison with its counterpart based on rectangular grids, the adaptive simplicial grid method is better suited to scenarios that involve multiple assets.

3 PROXIMITY MEASURES FOR PORTFOLIOS

To compare expected utility maximization with the MV method, Kroll *et al* (1984) employ the following approach: the expected utility is maximized along the MV efficient frontier, and the portfolio obtained from this maximization process is used for the comparison. We apply this technique to the PT optimization, ie, we solve the optimization problem (2.10).

Having computed the portfolios by solving the PT portfolio optimization problem (2.1) for CPT, and having obtained optimal portfolios by solving (2.10), one must now consider how to compare these portfolios. We require a "similarity" or "proximity" measure to accomplish this comparison.

Recall that λ^* denotes the optimal portfolio obtained by maximizing a CPT objective function according to (2.1), and λ^*_{MV} represents the MV portfolio in the comparison obtained by solving (2.10).

In the expected utility framework, Kroll *et al* (1984) employ the following objective functions ratio:

$$I_{\text{OBJR}} := \frac{\mathbb{E}[u(\xi^{\text{T}}\lambda^*_{\text{MV}})] - \mathbb{E}[u(\xi^{\text{T}}\lambda_{\text{naive}})]}{\mathbb{E}[u(\xi^{\text{T}}\lambda^*)] - \mathbb{E}[u(\xi^{\text{T}}\lambda_{\text{naive}})]},$$

where λ_{naive} is the "naive" portfolio with equal weights. This naive portfolio is introduced to avoid the dependence of the objective ratio on possible shifts in u (u and $u+C$ are equivalent with respect to ordering based on expected utility for any $C \in \mathbb{R}$). Observe that for the denominator the inequality $\mathbb{E}[u(\xi^{\text{T}}\lambda^*)] - \mathbb{E}[u(\xi^{\text{T}}\lambda_{\text{naive}})] \geqslant 0$ holds because λ^* is an optimal solution to the maximization problem (2.1), and λ_{naive} is a feasible solution to this problem. In applications of the above index, it is assumed that the denominator is strictly positive. The numerator can be negative, that is, it may happen that the optimal portfolio along the MV frontier has an objective value that is lower than that of the naive portfolio. Anyway, since λ^*_{MV} is a feasible solution of

(2.1), the inequality

$$I_{\text{OBJR}} \leqslant 1$$

clearly holds; in this inequality, larger values reflect greater proximity.

We note that the comparative study of DeMiguel *et al* (2009) reveals and emphasizes the validity and importance of the I_{OBJR} proximity index, as defined above.

We will employ this index as one of our comparison indexes, adapted to our case as follows:

$$I_{\text{OBJR}} = \frac{V(\xi^{\mathrm{T}}\lambda^*_{\text{MV}}) - V(\xi^{\mathrm{T}}\lambda_{\text{naive}})}{V(\xi^{\mathrm{T}}\lambda^*) - V(\xi^{\mathrm{T}}\lambda_{\text{naive}})}.$$

We assume the strict positivity of the denominator.

When our computational results are reported, the values of I_{OBJR} will be presented as percentages.

In the expected utility context, it is advantageous to use certainty equivalents for the comparison, because these equivalents are invariant under affine-linear transformations of u and have a direct economic interpretation (they are also called "cash equivalents"). Similar to the expected utility approach, in a PT context the certainty equivalent CE_v of a portfolio λ can be defined as follows:

$$v(\text{CE}_v[\xi^{\mathrm{T}}\lambda]) = V(\xi^{\mathrm{T}}\lambda) \Leftrightarrow \text{CE}_v[\xi^{\mathrm{T}}\lambda] = v^{-1}(V(\xi^{\mathrm{T}}\lambda)).$$

Recall from PT that v must be strictly monotonically increasing; thus, the inverse of v in the above equation must exist. This notion of a certainty equivalent has been employed for comparative purposes by various researchers (see, for example, De Giorgi and Hens 2009; Døskeland and Nordahl 2006).

Regarding expected utility, Pulley (1983) suggests utilizing the following ratio for the comparison:

$$\frac{\text{CE}_u[\xi^{\mathrm{T}}\lambda^*_{\text{MV}}]}{\text{CE}_u[\xi^{\mathrm{T}}\lambda^*]} = \frac{u^{-1}(\mathbb{E}[u(\xi^{\mathrm{T}}\lambda^*_{\text{MV}})])}{u^{-1}(\mathbb{E}[u(\xi^{\mathrm{T}}\lambda^*)])}. \tag{3.1}$$

Reid and Tew (1986) observe empirically for their data set that there is no substantial difference between I_{OBJR} and the above index.

Kallberg and Ziemba (1983) empirically compare the optimal portfolios obtained via expected utility maximization using different utility functions. This analysis is not focused on the comparison with MV. The proximity index is constructed as follows. Let λ^1 and λ^2 be optimal portfolios that are obtained via expected utility maximization according to two different utility functions. One of the two utility functions is selected as the reference utility; let us denote this utility function using u. Kallberg and Ziemba (1983) employ a normalized difference based on certainty equivalents as a proximity measure:

$$I_{\text{CER}} := \frac{\text{CE}_u[\xi^{\mathrm{T}}\lambda^1] - \text{CE}_u[\xi^{\mathrm{T}}\lambda^2]}{\text{CE}_u[\xi^{\mathrm{T}}\lambda^1]} = \frac{u^{-1}(\mathbb{E}[u(\xi^{\mathrm{T}}\lambda^1)]) - u^{-1}(\mathbb{E}[u(\xi^{\mathrm{T}}\lambda^2)])}{u^{-1}(\mathbb{E}[u(\xi^{\mathrm{T}}\lambda^1)])}.$$

In the PT case, because certainty equivalents are expressed in terms of net returns, they may have negative values, and the denominator in the above expression may be close or equal to zero. Therefore, in this paper we work with certainty equivalents in terms of gross returns. We use the above formula, based on the PT value function v:

$$I_{\mathrm{CER}} = \frac{\mathrm{CE}_v[\xi^{\mathrm{T}}\lambda^1] - \mathrm{CE}_v[\xi^{\mathrm{T}}\lambda^2]}{\mathrm{CE}_v[\xi^{\mathrm{T}}\lambda^1]},$$

where we assume that the denominator is positive. In the above expression, λ^1 is an optimal portfolio obtained by solving (2.1) in a CPT setting, whereas λ^2 is the optimal portfolio generated by MV. Consequently, we have the inequality

$$0 \leqslant I_{\mathrm{CER}},$$

and we observe that higher values of I_{CER} correspond to greater dissimilarity. Note the difference with respect to the index I_{OBJR}; for that metric, higher values indicate greater similarity.

As for I_{OBJR} above, we report our computational results for I_{CER} as percentages.

DeMiguel *et al* (2009) and De Giorgi and Hens (2009) utilize the difference between the certainty equivalents to assess the differences in the portfolio allocations that are produced according to PT and MV:

$$I_{\mathrm{CED}} := \mathrm{CE}_v[\xi^{\mathrm{T}}\lambda^*] - \mathrm{CE}_v[\xi^{\mathrm{T}}\lambda^*_{\mathrm{MV}}].$$

They interpret the difference as added value in monetary terms. We employ the following form of this index for comparing two portfolios:

$$I_{\mathrm{CED}} = \mathrm{CE}_v[\xi^{\mathrm{T}}\lambda^1] - \mathrm{CE}_v[\xi^{\mathrm{T}}\lambda^2],$$

where λ^1 and λ^2 are interpreted similarly to those for the index I_{CER}. In this situation, the inequality

$$0 \leqslant I_{\mathrm{CED}},$$

also holds, with higher values of I_{CED} indicating greater dissimilarity.

In our computational results, I_{CED} values will be reported as annualized returns in percentage terms (recall that monthly returns are provided in the original data set).

Simaan (1993) employs a proximity measure in the expected utility framework, which is different from, though quite similar to, I_{CED}. The author uses the optimization premium θ as a proximity index, which he defines as a solution of

$$\mathbb{E}[u(\xi^{\mathrm{T}}\lambda^*_{\mathrm{MV}} + \theta)] = \mathbb{E}[u(\xi^{\mathrm{T}}\lambda^*)].$$

Simaan (1993) interprets this quantity as the minimum certain net return the investor requires for investing in the "second-best" portfolio λ^*_{MV} instead of the optimal portfolio λ^*. Similar to the proximity measures discussed above, this measure is clearly invariant under positive linear transformations of the utility function u. We did not include this index in our computational study; rather, we used I_{CED} throughout.

TABLE 2 The indexes included in the monthly returns data set.

GSCITR	Goldman Sachs Commodity Index; total return
HFRIFFM	Hedge Fund Research International, Fund of Funds; market defensive index
I3M	Three-months US dollar Libor interest rate
JPMBD	JP Morgan Bond Index, Developed Markets; total return
MSEM	Morgan Stanley Emerging Markets Index; total return, stocks
MXWO	MSCI World Index; total return, stocks (developed countries)
NAREIT	FTSE, US Real Estate Index; total return
PE	LPX50, LPX Group Zurich, Listed Private Equities; total return

4 THE DATA SETS

The benchmark data set consists of monthly net returns for eight asset classes (indexes) over the February 1994–May 2011 period; it therefore includes a sample size of 208 elements. The indexes included in this data set are listed in Table 2. In the appendix (available online), Table A1 presents summary statistics for this data set and Table A2 provides the empirical correlation matrix.

In the following sections, we will repeatedly argue that the primary cause of the observed phenomenons is that the normality assumption does not hold in our case, meaning that our data set cannot be regarded as a sample from a normal distribution. To test the null hypothesis – that our data set can be considered a sample from a multivariate normal distribution – we performed Mardia's skewness and kurtosis tests for multivariate normality (for information regarding these tests, see, for example, Rencher (2002)). As shown in Table A3 in the appendix (available online), we obtained p-values of $p = 0.000$ for both multivariate skewness and multivariate kurtosis (actually, the p-values for both the skewness and the kurtosis were zeros within machine precision). Thus, the null hypothesis can clearly be rejected at a 99.9% significance level based on both tests.

We will work with three data sets in our numerical tests.

4.1 The first (original) data set

When we work with samples, all scenarios are equally likely. Because we wanted to explore the influence of probability weighting, we computed an empirical discrete distribution (a lottery) that included fifteen realizations. Because we sought to avoid distributional assumptions with respect to our data, we utilized k-means clustering with Manhattan distances. For information regarding k-means clustering, see, for example, Everitt *et al* (2011) and the references therein. The resulting empirical distribution is displayed in Table A4 in the appendix (available online). This empirical discrete distribution is our first data set, which we also call our original data set.

4.2 The second data set, involving a call option

The particular choice of $k = 15$ was motivated by our objective of obtaining a second data set by appending a European call option to the original data set. We needed to obtain a data set of this type for two reasons. First, in order to assess the method by Levy and Levy (2004), we wished to acquire a data set that involved a clear deviation from the normality assumption. Second, we also sought to test the CPT investors' preference for positive skewness, as discussed in Barberis and Huang (2008). We have appended a European call option on the index MXWO to the original data set. To generate this data set, we proceeded as follows.

We considered a given data set consisting of scenarios and corresponding probabilities to be an incomplete market that should be arbitrage-free to allow for the incorporation of a call option. To test this condition, we first computed state prices by solving the following linear programming problem:

$$\left.\begin{aligned} \max_{\pi,\varepsilon} \quad & \varepsilon, \\ \sum_{s=1}^{S} \pi_s &= 1, \\ \sum_{s=1}^{S} r_s^k \pi_s &= r_{\mathrm{f}}, \quad k = 1, \dots, K, \\ \pi_s &\geqslant \varepsilon, \quad s = 1, \dots, S, \end{aligned}\right\} \tag{4.1}$$

where r_{f} is the risk-free rate. Let (π^*, ε^*) be an optimal solution to the above problem. This problem allows for the computation of state prices, with the smallest state price being maximal. The market is arbitrage-free if $\varepsilon^* > 0$ holds for the optimal solution. For information regarding state prices and incomplete markets, see, for example, Magill and Quinzii (1996) and Černý (2009).

In our situation, we had $K = 8$ assets and sought to generate an arbitrage-free data set for the monthly risk-free rate $r_{\mathrm{f}} = 0.002$. To achieve this goal, we generated empirical discrete distributions by applying an increasing number of scenarios S with k-means clustering ($k = S$) to our benchmark data set consisting of 208 elements. For each of these empirical distributions, we solved (4.1). Finally, for $S = 15$, we obtained positive state prices. This empirical discrete distribution with $S = 15$ realizations and $K = 8$ assets served as our first data set, as discussed in Section 4.1.

Using the state prices obtained by solving (4.1), we add a call option on the sixth index, MXWO, to our data set. The data matrix is supplemented with a column that corresponds to this call option, with the entries scaled such that for the added column $\hat{r}$, the relation $\sum_{s=1}^{S} \hat{r}_s \pi_s^* = r_{\mathrm{f}}$ holds, guaranteeing the arbitrage-free nature of the new data set. As a strike price (in terms of net returns), we choose 0.1, which ensures there is a positive payoff in only one state, namely $s = 12$ (see Table A4 in the online appendix). Thus, in the added column $\hat{r}$, the only nonzero element is $\hat{r}_{12} = 0.283$, with the specific value computed as discussed above. Consequently, apart from the distributional assumptions, our setting is quite similar to the context examined by Barberis and Huang (2008).

4.3 The third data set, related to a normal distribution

For comparative purposes, we also generated a third data set for use in testing the effects of a normality assumption, meaning the assumption that the data set has been constructed by sampling and subsequent clustering of normally distributed returns. This data set was generated as follows. Using the empirical expected value vector μ and the empirical covariance matrix Σ from the original scenarios, we simulated a sample of 10 000 elements from the corresponding multivariate normal distribution by employing the standard method based on the Cholesky-factorization of Σ. Subsequently, we employed k-means clustering to obtain fifteen scenarios and their corresponding probabilities.

To cross-check the quality of our generated sample and the results of Mardia's multivariate normality test, we performed Mardia's skewness and kurtosis tests (see, for example, Rencher 2002) on the sample to test the null hypothesis that the sample can be regarded as being from a multivariate normal distribution. For the sample with 10 000 elements, we obtained the following results. The skewness test yielded a p-value of $p_{\mathrm{Skew}} = 0.85$, and the kurtosis test produced $p_{\mathrm{Kurt}} = 0.31$. We also performed the tests with the first 208 sample elements and obtained $p_{\mathrm{Skew}} = 0.80$ and $p_{\mathrm{Kurt}} = 0.17$, as shown in Table A3 in the appendix (available online). Thus, for both sample sizes, based on both the skewness and the kurtosis tests, we cannot reject the null hypothesis, even at a 90% level of significance.

4.4 The data set for the CPT parameter settings

For the parameters of the PT value function and the probability weighting functions, we utilized the settings published by Abdellaoui *et al* (2007). In Appendix C of their paper, the authors present the results of an experimental elicitation of the PT parameters for forty-eight subjects. The reference point is 0 throughout. Several parameter values are presented for the different definitions of loss aversion. For our numerical experiments, we chose the classical Kahneman–Tversky definition of the loss-aversion coefficient.

With respect to probability distortion, Abdellaoui *et al* (2007) present probabilities p_{g} and p_{l} for gains and losses, respectively, with the properties of $w^{+}(p_{\mathrm{g}}) = 0.5$ and $w^{-}(p_{\mathrm{l}}) = 0.5$. To determine the parameters δ and γ in the probability distortion functions (2.9), the corresponding equations must be solved under a fixed probability p for the parameters δ and γ. These equations have a unique solution for $p \geqslant 0.5$, whereas for $p < 0.5$ we have computed the solutions with the lowest absolute deviations in the equations. We employed the straightforward grid search approach in determining δ and γ, as there is no need for a high-precision solution.

In this paper, we present our comparative numerical results, taking all forty-eight of the settings into account. We consider the subjects participating in the experiment as separate investors.

5 CPT–MV: COMPARING THE OPTIMAL PORTFOLIOS

In Section 5.1, we present our basic comparative numerical results. In Section 5.2, we provide an intuitive explanation of the observed differences.

5.1 Main comparative results

For the original data set, the data set with an added call option and the data set based on an associated normal distribution, we proceeded as follows. We have solved the portfolio optimization problem (2.1) for the CPT objective function. Subsequently, we maximized the CPT objective function along the MV frontier. All the computations were performed in turn for all the subjects (which are considered investors) by utilizing the PT value-function parameters that are listed in the paper by Abdellaoui *et al* (2007).

Next, we used the proximity measures I_{OBJR}, I_{CED} and I_{CER}, which are defined and discussed in Section 3, to compare the portfolios obtained. As discussed in Section 3, the values of all three proximity indexes will be reported in percentage terms; I_{OBJR} and I_{CER} represent ratios, whereas I_{CED} is expressed in terms of annual returns. These results are presented in Table 3.

In addition to Table 3, Figure 2 displays the comparative results for the three data types. In this figure, the filled circles represent expected values, and the upward- and downward-oriented vertical line segments represent the upper and lower semi-deviations, respectively. The distributions of the proximity indexes are clearly asymmetric (see Table 3); therefore, we believe figures of this type provide additional intuitive information regarding the differences across the data sets.

TABLE 3 Summary statistics for the proximity indexes in the CPT–MV comparison.

(a) CPT–MV; I_{OBJR} index

	Original	Call option	Normal approximation
Minimum	−18.46	−21.99	99.66
Mean	76.89	58.48	99.99
Median	97.07	73.69	100.0
Maximum	99.43	97.70	100.0
Standard deviation	38.15	36.40	0.049
Lower semi-deviation	33.93	29.98	0.048
Upper semi-deviation	17.43	20.65	0.008
Skewness	−1.544	−0.906	−6.616
Kurtosis	3.604	2.396	44.85

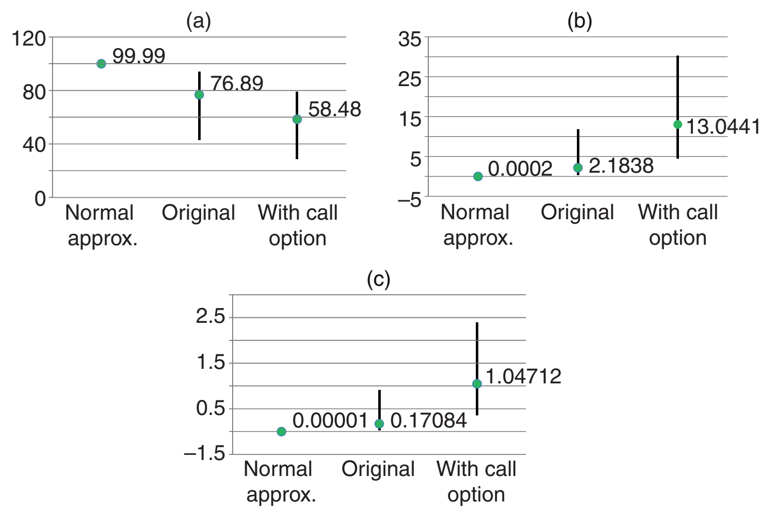
(b) CPT–MV; I_{CED} index

	Original	Call option	Normal approximation
Minimum	0.0028	0.034	−0.001
Mean	2.1838	13.04	0.0002
Median	0.1734	5.238	0.000
Maximum	67.87	83.72	0.006
Standard deviation	9.877	19.26	0.001
Lower semi-deviation	1.874	8.587	0.0004
Upper semi-deviation	9.698	17.24	0.0009
Skewness	6.265	2.184	4.590
Kurtosis	41.61	7.325	28.16

(c) CPT–MV; I_{CER} index

	Original	Call option	Normal approximation
Minimum	0.000	0.003	−0.00009
Mean	0.171	1.047	0.00001
Median	0.014	0.435	0.000
Maximum	5.151	6.525	0.0005
Standard deviation	0.751	1.508	0.00008
Lower semi-deviation	0.146	0.685	0.00003
Upper semi-deviation	0.737	1.343	0.00007
Skewness	6.207	2.104	4.594
Kurtosis	41.06	6.987	28.19

FIGURE 2 An examination of the indexes I_{OBJR}, I_{CED} and I_{CER} across the three data sets for the CPT–MV comparison.



(a) CPT–MV comparison; I_{OBJR} index. (b) CPT–MV comparison; I_{CED} index. (c) CPT–MV comparison; I_{CER} index.

Recall that the maximal value for I_{OBJR} is 100%, with larger values representing greater proximity. By contrast, for the other two indexes, I_{CED} and I_{CER}, larger values correspond to greater degrees of dissimilarity between the evaluated portfolios.

For the results obtained with the original data set, we observe a substantial difference between those using the CPT optimization and those using the CPT(MV) approach. We assessed the relationship between the CPT-optimal portfolios and the portfolios obtained by maximizing the CPT objective function along the MV frontier in terms of annual returns. On average, in this comparison, there is a remarkable $I_{\text{CED}} = 2.2\%$ deviation with respect to the certainty equivalents for the assessed alternatives, with a maximum value of $I_{\text{CED}} = 67.9\%$, a minimal value of $I_{\text{CED}} = 0.003\%$, a standard deviation of 9.9% and positive skewness.

This conclusion is supported by the following fact: if we restrict our attention to a middle range of the obtained I_{CED} values, we observe that for twenty-five investors (more than 50% of the investors) the I_{CED} values are between 0.05% and 0.78%, with an average of 0.26%. For annualized returns, this is a clear indication of the differences in the results obtained using the two approaches. At first glance, these numbers appear to be rather small, but they are quite large in the wealth management field (between 5bps and 78bps); for an explanation of this fact, see De Giorgi and Hens (2009, Section 2).

Next, we discuss our computational results for the data set supplemented with a European call option. The effects discussed above are magnified based on the second data set. In terms of annual returns, we now have a substantial average deviation of $I_{\text{CED}} = 13.0\%$, along with a maximum value of $I_{\text{CED}} = 83.7\%$, a minimal value of $I_{\text{CED}} = 0.03\%$, a standard deviation 19.3% and positive skewness. If a middle range of twenty-five investors is again chosen, we obtain $1.9\% < I_{\text{CED}} < 25.4\%$ for the I_{CED} values, with an average I_{CED} of 8.8%. This result indicates a very great difference between the optimal portfolios obtained via direct CPT optimization and the portfolios computed by optimization along the MV frontier.

Although the differences between the results obtained using these two approaches are clear, the question of what factors cause this phenomenon must be addressed. This issue will be discussed in Section 5.2, where we identify the preference of CPT investors for positive skewness as one of the causes for the difference.

Finally, for the third data set with the associated normal distribution, we note that the difference between the CPT and CPT(MV) is greatly diminished. With respect to annualized returns, we now have an average deviation of certainty equivalents of $I_{\text{CED}} = 0.0002\%$, a maximal value of $I_{\text{CED}} = 0.006\%$, a minimal value of $I_{\text{CED}} = -0.001\%$ and a standard deviation of 0.001%. The reason is that the CPT-optimal portfolios are now located on or very close to the efficient MV frontier.

The values obtained for the other two proximity indexes I_{OBJR} and I_{CER} also clearly support our conclusions; see Table 3 and Figure 2.

Since the Sharpe ratio is a widely used performance measure, for comparative purposes we also computed the Sharpe ratios for our portfolios obtained from the CPT and CPT(MV) approaches. For this comparison we selected $r_{\text{f}} = 0.0015$ as the risk-free rate (recall that we have monthly data). The summary statistics of the results are displayed in Table 4.

For both the original data set and the data set related to a normal distribution, we observe a negligible difference between the Sharpe ratios obtained by the CPT and CPT(MV) approaches. For the data set with the call option, we observe Sharpe ratios for the portfolios obtained using the CPT(MV) approach that are substantially higher than those obtained using the CPT approach.

5.2 What are the causes of the observed differences?

As mentioned above, Barberis and Huang (2008) proved that, under appropriate assumptions, CPT-based investors prefer positively skewed securities. In this section, we discuss our numerical results from this point of view. Skewness is measured by the coefficient of skewness $\gamma_1 := \mu_3/\sigma^3$ of the random portfolio returns, with μ_3 denoting the third central moment and σ denoting the standard deviation.

TABLE 4 Summary statistics for the Sharpe ratios of the portfolios obtained by the CPT and CPT(MV) approaches.

(a) CPT approach

	Original	Call option	Normal distribution
Minimum	−0.016	0.041	0.046
Mean	0.139	0.058	0.520
Median	0.116	0.044	0.661
Maximum	0.329	0.265	0.668
Standard deviation	0.099	0.042	0.256

(b) CPT–MV approach

	Original	Call option	Normal distribution
Minimum	0.008	0.007	0.047
Mean	0.145	0.139	0.516
Median	0.117	0.115	0.660
Maximum	0.344	0.344	0.668
Standard deviation	0.122	0.123	0.256

Our observations, related to the results using the original data set, are as follows. Regarding the positive skewness preferences of the CPT investors, we should note that Figure 3 displays the skewness of the optimal portfolios that were obtained using the CPT optimization approach and the portfolios that were computed using the CPT(MV) method. It can be observed that, compared with the CPT(MV) portfolios, the CPT-optimal portfolios are less negatively skewed and may even exhibit positive skewness.

Next, we examine the results with the data set that included the call option. Figure 4 displays the skewness values of the optimal portfolios that were obtained using the two different approaches. This figure clearly indicates that CPT-based investors exhibit a definitive preference for positive skewness. Indeed, twenty-five of the investors do not diversify but invest the entirety of their wealth in the call option, and fifteen additional investors invest a positive fraction of their wealth in this security. By contrast, in the CPT(MV) case, none of the investors invest more than the negligible fraction of 0.006 of their initial wealth in the call option. Thus, our numerical results completely reflect the preference of CPT investors for positively skewed securities, which is also demonstrated by Barberis and Huang (2008) under a normality assumption. Note that

FIGURE 3 The skewness of the optimal portfolios that have been obtained via the CPT and CPT(MV) approaches with the original data set.

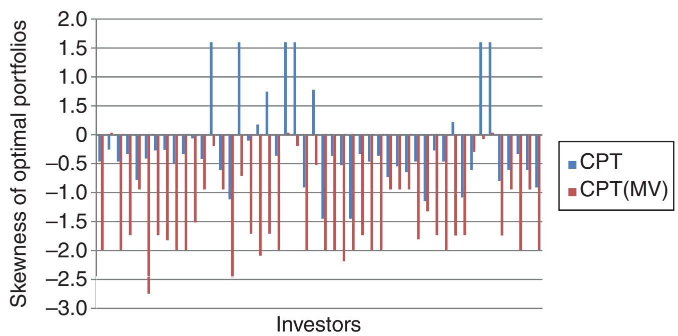
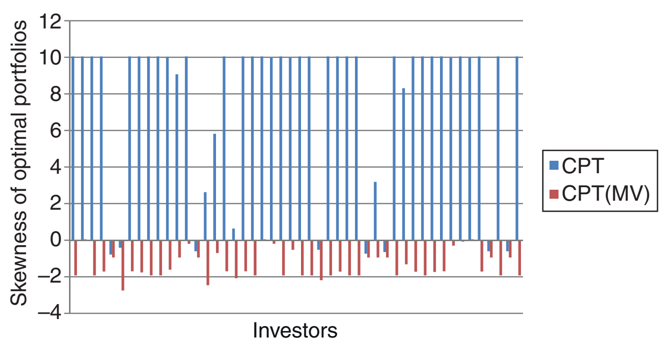


FIGURE 4 The skewness of the optimal portfolios obtained via the CPT and CPT(MV) approaches with the addition of a call option.



in this situation the European call option serves as a lottery-like security that produces a positive payoff only in a single state of the world. We emphasize that this call option has a sizable positive skewness of 10.1.

Barberis and Huang (2008) identify probability weighting as the main cause of the skewness-loving behavior of CPT-based investors. To assess whether probability weighting is the only cause of this preference for positive skewness, we performed a test in which we set the probability weighting parameters as $\gamma = \delta = 1$. In this case, CPT reduces to PT without probability weighting. The results of this assessment are displayed in Figure 5.

FIGURE 5 The skewness of the optimal portfolios obtained via the CPT and CPT(MV) approaches, with the call option added but without probability weighting.

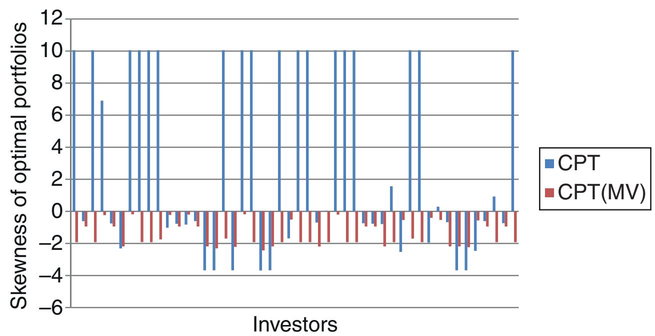


TABLE 5 Elapsed time in seconds for solving CPT optimization problems, with n denoting the number of assets.

n	8	9	10	11	12	13	14
t elapsed	6.7	12.9	23.6	42.7	74.5	127.1	227.7
n	15	16	17	18	19	20	
t elapsed	365.5	567.0	854.2	1291.8	1899.1	2782.4	

Computing environment: Windows 7, 64 bit, 2.7 GHz processor and 8.0 GB RAM.

In comparing Figure 5 and Figure 4, we observe that the degree of skewness-loving is reduced in Figure 5 but still exists. Thus, other aspects of CPT besides probability weighting play a role in the skewness-loving tendencies of CPT investors.

Therefore, on the basis of our numerical results, we identify the preference of CPT investors for positively skewed securities as one of the main reasons for obtaining different results when applying either the CPT or the CPT(MV) approach.

5.3 Computational costs

In terms of computational time, the results show that it is much more expensive to solve the CPT optimization problem than it is to optimize the CPT objective function along the MV frontier. The computational times in seconds are displayed in Table 5, where for dimensions 8 and 9 we took the average computational time over forty-eight investors. For the dimensions from 10 to 20, we generated discrete distributions with fifteen realizations for monthly asset returns of assets included in the DJIA index, just for recording the computational times. From twenty-two assets upward, the computational time exceeds one hour.

In contrast, optimizing CPT along the MV frontier took less than 0.13 seconds for all of the above cases; this includes the computing time for generating the portfolios along the MV frontier.

6 CONCLUSION

To the best of our knowledge, our paper proposes the first general algorithm for computing asset allocations for CPT. This issue is numerically difficult to address but highly relevant to the field of finance. Using the algorithm, we performed a numerical study based on a real-life data set and two variants of this data set to numerically test the differences between the PT approach and MV analysis.

Except for the data set related to normally distributed returns, we observe that the CPT model differs substantially from MV analysis in terms of the results they generate. As one of the reasons for this difference, we identify the preference of CPT investors for positively skewed securities, and we numerically verify this preference by adding a call option to our original data set.

For the data set obtained by sampling and subsequent clustering from normally distributed returns, we observed that the difference between the CPT and the CPT(MV) approaches diminishes. Since in our experiments the CPT(MV) approach turned out to be numerically much faster than the CPT approach (see the previous section), future research should investigate whether the theoretical results regarding the equality of the CPT and CPT(MV) approaches obtained under the assumption of normally distributed returns can be extended to empirical discrete distributions corresponding to an underlying normal distribution.

In addition, future research should use out-of-sample data to compare the three asset-allocation models examined. In particular, it would be interesting to investigate whether the simple $1/n$ rule that outperforms the MV model out of sample can also outperform the asset allocations derived from PT.

Within the framework of the CAPM, Levy (2012) presents a thorough comparative analysis and discussion of the MV and CPT approaches. It would be interesting to check the robustness of these results with respect to empirical distributions.

DECLARATION OF INTEREST

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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